On a canonical lattice structure on the effect algebra of a von Neumann algebra

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14.12.2005

Abstract

Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{R}_{sa}$ be the set of hermitean (i.e. selfadjoint) elements of $\mathcal{R}$. It is well known that $\mathcal{R}_{sa}$ is a lattice with respect to the usual partial order $\leq$ if and only if $\mathcal{R}$ is *abelian*. We define and study a new partial order on $\mathcal{R}_{sa}$, the spectral order $\leq_s$, which extends $\leq$ on projections, is coarser than the usual one, but agrees with it on abelian subalgebras, and turns $\mathcal{R}_{sa}$ into a *boundedly complete lattice*. The effect algebra $\mathcal{E}(\mathcal{R}) := \{A \in \mathcal{R}_{sa} | 0 \leq A \leq I\}$ is then a complete lattice and we show that the mapping $A \mapsto R(A)$, where $R(A)$ denotes the range projection of A, is a homomorphism from the lattice $\mathcal{E}(\mathcal{R})$ onto the lattice $\mathcal{P}(\mathcal{R})$ of projections if and only if $\mathcal{R}$ is a finite von Neumann algebra.

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1 Introduction

In this paper $\mathcal{R}$ is a von Neumann algebra contained in the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators of the Hilbert space $\mathcal{H}$, $\mathcal{R}_{sa}$ denotes the set of hermitean elements of $\mathcal{R}$ and $\mathcal{E}(\mathcal{R})$ the *effect algebra of* $\mathcal{R}$, i.e. set of all positive operators in $\mathcal{R}$ less or equal to I. This is meant with respect to the usual partial order on $\mathcal{R}_{sa}$:

$$A \leq B \quad \text{if and only if} \quad \forall\, x \in \mathcal{H} : < Ax, x > \leq < Bx, x > .$$

Definition 1.1 *A* lattice *is a partially ordered set* $(\mathbb{L}, \leq)$ *such that any two elements* $a, b \in \mathbb{L}$ *possess a* maximum $a \vee b \in \mathbb{L}$ *and a* minimum $a \wedge b \in \mathbb{L}$.
Let $\mathfrak{m}$ *be an infinite cardinal number.*
The lattice $\mathbb{L}$ *is called* $\mathfrak{m}$*-complete, if every family* $(a_i)_{i\in I}$ *has a supremum* $\bigvee_{i\in I} a_i$ *and an infimum* $\bigwedge_{i\in I} a_i$ *in* $\mathbb{L}$*, provided that* $\#I \leq \mathfrak{m}$ *holds. A lattice* $\mathbb{L}$ *is simply called complete, if every family* $(a_i)_{i\in I}$ *in* $\mathbb{L}$ *(without any restriction of the cardinality of* I*) has a maximum and a minimum in* $\mathbb{L}$.
$\mathbb{L}$ *is said to be boundedly complete if every bounded family in* $\mathbb{L}$ *has a maximum and a minimum.*
If a lattice has a zero element 0 *(i.e.* $\forall a \in \mathbb{L} : 0 \leq a$*) and a* unit element 1 *(i.e.* $\forall a \in \mathbb{L} : a \leq 1$*) then completeness and bounded completeness are the same.*
Note that a complete lattice always has a zero and a unit element, namely $0 := \bigwedge_{a\in\mathbb{L}} a$ *and* $1 := \bigvee_{a\in\mathbb{L}} a$.
A lattice $\mathbb{L}$ *is called* distributive *if the two distributive laws*

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

hold for all elements $a, b, c \in \mathbb{L}$.

$\bigvee_{i\in I} a_i$ is characterized by the following universal property:

(i) $\forall j \in I : \quad a_j \leq \bigvee_{i\in I} a_i$

(ii) $\forall c \in \mathbb{L} : \quad ((\forall i \in I : a_i \leq c) \Rightarrow \bigvee_i a_i \leq c)$.

An analogous universal property characterizes the minimum $\bigwedge_i a_i$.

Note that if $\mathbb{L}$ is a distributive complete lattice, then in general

$$a \wedge (\bigvee_{i\in I} b_i) \neq \bigvee_{i\in I} (a \wedge b_i),$$

so completeness and distributivity do not imply *complete distributivity*!

Now a well known theorem states ([6],p.186):

Theorem 1.1 *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}_{sa}$ is a lattice with respect to the partial order $\leq$ if and only if $\mathcal{A}$ is abelian.*

But note:

(i) The relation $\leq$ is closely connected with the linear structure of $\mathcal{A}_{sa}$: $A \leq B$ if and only if $B - A \geq 0$, whereas

(ii) $A \geq 0$ can also be characterized by the fact that the spectrum $sp(A)$ of A is contained in $\mathbb{R}_+$, the set of nonnegative real numbers.

In this paper we will, based on the foregoing observation, define a new partial order $\leq_s$, called the **spectral order** on $\mathcal{R}_{sa}$, and study its main properties.

After publishing the first version of this paper in the arXiv, David Sherman ([10]) informed me that the definition of the spectral order and its main properties are already contained in a paper of M.P. Olson ([9]). Because I came to these results in a more general context ([2]), and because the first version contains an important application of the spectral order, I think that it is justified to publish this second version.

The spectral order can be defined by elementary relations between the spectral projections of the operators in question. The spectral order agrees (by its very definition) on projections with the usual one but, in general, it is coarser than that. This means that $A \leq_s B$ implies $A \leq B$ but not vice versa. It turns out that the two partial orders agree for all commuting pairs of hermitean operators A, B, a fact that should be important for possible applications in quantum physics. In section 3 we show that the spectral order turns $\mathcal{R}_{sa}$ into a boundedly complete lattice $(\mathcal{R}_{sa}\vee_s, \wedge_s)$. This is equivalent to the completeness of the sublattice $\mathcal{E}(\mathcal{R})$. There is a natural hull operation on the effect algebra $\mathcal{E}(\mathcal{R})$:

$$\begin{aligned} R : \mathcal{E}(\mathcal{R}) &\rightarrow \mathcal{P}(\mathcal{R}) \\ A &\mapsto R(A), \end{aligned}$$

where $R(A)$ denotes the range projection of A. R always respects the join $\vee_s$. It respects also the meet $\wedge_s$ if and only if $\mathcal{R}$ is a *finite* von Neumann algebra. A similar result has been obtained by C. Cattaneo and J. Hamhalter in [1] - though for the usual order, where $\vee$ and $\wedge$ are only partially defined operations.

2 The spectral order

Let $\mathcal{P}(\mathcal{R})$ be the lattice of projections in the von Neumann algebra $\mathcal{R}$ and let $\mathcal{P}_0(\mathcal{R}) := \mathcal{P}(\mathcal{R}) \setminus \{0\}$. We have introduced in [2] the notion of *observable function* of an element $A \in \mathcal{R}_{sa}$. This is a bounded real valued function f_A on the space $\mathcal{D}(\mathcal{R})$ of all *dual ideals* of the lattice $\mathcal{P}(\mathcal{R})$, defined by

$$\forall\ \mathcal{J} \in \mathcal{D}(\mathcal{R}) :\ f_A(\mathcal{J}) := \inf\{\lambda \in \mathbb{R} \mid E^A_\lambda \in \mathcal{J}\},$$

where $(E^A_\lambda)_{\lambda\in\mathbb{R}}$ is the spectral family of A. The restriction of f_A to the set $\mathcal{D}_{pr}(\mathcal{R})$ of all *principal dual ideals* $H_P := \{Q \in \mathcal{P}(\mathcal{R}) \mid Q \geq P\}$, $(P \in \mathcal{P}_0(\mathcal{R}))$, defines a function

$$\begin{array}{cccc} r_A : & \mathcal{P}_0(\mathcal{R}) & \to & \mathbb{R} \\ & P & \mapsto & f_A(H_P). \end{array}$$

The functions $r : \mathcal{P}_0(\mathcal{R}) \to \mathbb{R}$ that are induced by observable functions are characterized by the property that

$$r(\bigvee_{k\in\mathbb{K}} P_k) = \sup_{k\in\mathbb{K}} r(P_k)$$

holds for all families $(P_k)_{k\in\mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$. Therefore, they are called *completely increasing functions* ([2]).

The following result is easy to prove:

Proposition 2.1 *Let $A, B \in \mathcal{R}_{sa}$ with spectral families E^A and E^B, respectively. Then*

$$r_A \leq r_B \quad \textit{if and only if} \quad \forall\ \lambda \in \mathbb{R} :\ E^B_\lambda \leq E^A_\lambda.$$

One can reconstruct f_A from r_A, because

$$\forall\ \mathcal{J} \in \mathcal{D}(\mathcal{R}) :\ f_A(\mathcal{J}) = \inf_{P\in\mathcal{J}} r_A(P).$$

The spectral family of $P \in \mathcal{P}(\mathcal{R})$ is given by

$$E^P_\lambda = \begin{cases} 0 & \text{for } \lambda < 0 \\ I - P & \text{for } 0 \leq \lambda < 1 \\ I & \text{for } 1 \leq \lambda. \end{cases}$$

If $P, Q \in \mathcal{P}(\mathcal{R})$, then $P \leq Q$ if and only if $I - Q \leq I - P$ i.e.

$$P \leq Q \quad \text{if and only if} \quad \forall\ \lambda \in \mathbb{R} :\ E^Q_\lambda \leq E^P_\lambda.$$

These simple facts lead us to the following basic

Definition 2.1 *Let $A, B \in \mathcal{R}_{sa}$ with corresponding spectral families $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ and $E^B = (E^B_\lambda)_{\lambda\in\mathbb{R}}$, respectively. Then $A \leq_s B$ if and only if*

$$\forall\ \lambda \in \mathbb{R} : \ E^B_\lambda \leq E^A_\lambda.$$

$\leq_s$ is a partial order on $\mathcal{R}_{sa}$. It is called the **spectral order**.

Remark 2.1 *The mapping $A \mapsto r_A$ from $\mathcal{R}_{sa}$ onto the set of completely increasing functions is not additive. Therefore we can not expect that $\leq_s$ is a linear order. Hence the spectral order should be different from the usual one.*

In the sequel we will investigate the relations between the spectral order and the usual order on $\mathcal{R}_{sa}$. To this end we show that we can confine ourselves to the subset $\mathcal{E}(\mathcal{R})$ of hermitean operators between 0 and I. This makes the discussion somewhat more comfortable.

Lemma 2.1 *Let $a, b \in \mathbb{R}, a > 0$. Then for all $A, B \in \mathcal{R}_{sa}$ the following equivalences hold:*

1. $A \leq B \quad \Longleftrightarrow \quad aA + bI \leq aB + bI,$
2. $A \leq_s B \quad \Longleftrightarrow \quad aA + bI \leq_s aB + bI.$

Proof: The first equivalence is trivial. The second follows from the simple fact that the spectral family E^{aA+bI} of $aA + bI$ is given by

$$E^{aA+bI}_\lambda = E^A_{a^{-1}\lambda-b},$$

where E^A is the spectral family of A:

$$E^{aB+bI}_\lambda = E^B_{a^{-1}\lambda-b} \leq E^A_{a^{-1}\lambda-b} = E^{aA+bI}_\lambda.$$

□

The following example (which is taken from [4], p.251) shows that the two partial orders on $\mathcal{R}_{sa}$ are different.

Remark 2.2 *Let $\mathcal{H} = \mathbb{C}^2$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $P \leq A$, but $P \not\leq_s A$.*

Proof: A simple calculation shows $P \leq A$. A has eigenvalues $\lambda_1 = \frac{3}{2} - \frac{1}{2}\sqrt{5} < 1$ and $\lambda_2 = \frac{3}{2} + \frac{1}{2}\sqrt{5} > 1$. Therefore $E^A_{\lambda_1}$ is the projection onto the line $\mathbb{C}\begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$, but $E^P_{\lambda_1}$ is the projection onto the line $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence $E^A_{\lambda_1} \not\leq E^P_{\lambda_1}$, i.e. $P \not\leq_s A$. □

Theorem 2.1 *The spectral order on $\mathcal{R}_{sa}$ is coarser than the usual one, i.e.*

$$\forall\ A, B \in \mathcal{R}_{sa}:\ (A \leq_s B \implies A \leq B).$$

Proof: According to 2.1 we can assume that $A, B \in \mathcal{E}(\mathcal{R})$. By the spectral theorem A and B are (in norm) arbitrarily close to

$$\sum_{k=1}^{n} \frac{k}{n}(E^A_{\frac{k}{n}} - E^A_{\frac{k-1}{n}})$$

and

$$\sum_{k=1}^{n} \frac{k}{n}(E^B_{\frac{k}{n}} - E^B_{\frac{k-1}{n}})$$

respectively if n is chosen sufficiently large. Because of $-E^A_\lambda \leq -E^B_\lambda$ for all λ we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k}{n}(E^A_{\frac{k}{n}} - E^A_{\frac{k-1}{n}}) &= I - \frac{1}{n}(E^A_{\frac{n-1}{n}} + E^A_{\frac{n-2}{n}} + \ldots + E^A_0) \\
&\leq I - \frac{1}{n}(E^B_{\frac{n-1}{n}} + E^B_{\frac{n-2}{n}} + \ldots + E^B_0) \\
&= \sum_{k=1}^{n} \frac{k}{n}(E^B_{\frac{k}{n}} - E^B_{\frac{k-1}{n}}).
\end{aligned}$$

Hence $A \leq B$. $\square$

Corollary 2.1 *If $A, B \in \mathcal{R}_{sa}$ commute, then*

$$A \leq_s B \iff A \leq B.$$

Proof: The spectral projection E^A_λ is the projection onto the kernel of $(A - \lambda I)^+$. Therefore, if $A \leq B$ and if A and B commute, it follows that

$$\forall\ \lambda:\ (A - \lambda I)^+ \leq (B - \lambda I)^+.$$

Hence $ker(B - \lambda I)^+ \subseteq ker(A - \lambda I)^+$, i.e. $E^B_\lambda \leq E^A_\lambda$ for all λ. $\square$

Remark 2.3 *If $A, B \in \mathcal{R}_{sa}$, then $A \leq_s B$ does not imply that A and B commute: Let $B \in \mathcal{E}(\mathcal{R})$ be invertible, P an arbitrary projection. Then $aP \leq_s B$ for sufficiently small $a > 0$, but $aPB \neq BaP$ in general.*

Remark 2.4 *The proof of the foregoing corollary shows the core of the difference between the two partial orders: if A and B are two* **noncommuting** *hermitean operators, then $A \leq B$ does* **not** *imply the relation $A^+ \leq B^+$ for their positive parts. The reason for this is that the function $t \mapsto t^2$ is not operator-monotonic. Indeed T. Ogasawara has shown ([8], see also [3], theorem 7.3.4) that a C^*-algebra $\mathcal{A}$ with the property*

$$\forall\ a, b \in \mathcal{A} :\ (0 \leq a \leq b \quad \Longrightarrow \quad a^2 \leq b^2)$$

is necessarily abelian.

Corollary 2.2 *Let $A, B \in \mathcal{E}(\mathcal{R})$ such that A or B is a projection. Then*

$$A \leq_s B \iff A \leq B.$$

Proof: According to corollary 2.1 we only have to show that $A \leq B$ forces A and B to commute. We may assume that A is a projection P, because $I - B \leq I - A$ reduces the other possibility to the first one.
Now $P \leq B$ implies

$$P \leq PBP \leq PIP = P,$$

i.e.

$$P = PBP.$$

Therefore B leaves imP invariant: Let x be a unit vector from imP. Then we can write $Bx = y + z$ with $y \in imP,\ z \in (imP)^\perp$. Because of $|y|^2 + |z|^2 = |Bx|^2 \leq 1$ and

$$x = Px = PBx = y,$$

$z = 0$ follows. As B is hermitean, $(imP)^\perp$ is B-invariant, too. This shows $PB = BP$, and from $P = PBP$ we even get $P = PB = BP$. $\Box$

Note that the example in Remark 2.2 shows that the assumption $A, B \in \mathcal{E}(\mathcal{R})$ is essential in the foregoing corollary.

We close this section with a short comment on a possible physical interpretation of the spectral order.
Let E^A be the spectral family of $A \in \mathcal{E}(\mathcal{R})$. Then

$$I - E^A_\lambda = \chi_{]\lambda,1]}(A)$$

where $\chi_{]\lambda,1]}$ denotes the characteristic function of the interval $]\lambda, 1]$. Hence

$$\forall\ A, B \in \mathcal{E}(\mathcal{R}) :\ (A \leq_s B \quad \Longleftrightarrow \quad \forall\ \lambda \in [0,1] :\ \chi_{]\lambda,1]}(A) \leq \chi_{]\lambda,1]}(B).$$

If x is a unit vector in $\mathcal{H}$ then $< \chi_{]\lambda,1]}(A)x, x >$ is usually interpreted as the probability that measuring the observable A in the pure state x gives a result lying in the interval $]\lambda, 1]$.

3 The spectral lattice

In this section we show that $\mathcal{R}_{sa}$ is a boundedly complete lattice with respect to the spectral order. In order to motivate our definitions we reconsider the lattice operations for projections.
If $P, Q \in \mathcal{P}(\mathcal{R})$ then

$$I - (P \vee Q) = (I - P) \wedge (I - Q) \quad \text{and} \quad I - (P \wedge Q) = (I - P) \vee (I - Q).$$

Therefore the spectral families of $P \vee Q$ and $P \wedge Q$ are given by

$$E_\lambda^{P \vee Q} = E_\lambda^P \wedge E_\lambda^Q$$

and

$$E_\lambda^{P \wedge Q} = E_\lambda^P \vee E_\lambda^Q$$

respectively. This leads to the following generalization.

Proposition 3.1 *Let $\mathfrak{E} = (E_\lambda)_{\lambda \in \mathbb{R}}$ and $\mathfrak{F} = (F_\lambda)_{\lambda \in \mathbb{R}}$ be spectral families in $\mathcal{R}$. Then*

(i) $(\mathfrak{E} \vee \mathfrak{F})_\lambda := E_\lambda \wedge F_\lambda \qquad (\lambda \in \mathbb{R})$ *and*

(ii) $(\mathfrak{E} \wedge \mathfrak{F})_\lambda := \bigwedge_{\mu > \lambda}(E_\mu \vee F_\mu) \qquad (\lambda \in \mathbb{R})$

define spectral families $\mathfrak{E} \vee \mathfrak{F}$ and $\mathfrak{E} \wedge \mathfrak{F}$ respectively in $\mathcal{R}$.

Proof: The only not totally trivial point is the continuity of $\mathfrak{E} \wedge \mathfrak{F}$ from the right:

$$\begin{aligned} \bigwedge_{\nu > \lambda} (\mathfrak{E} \wedge \mathfrak{F})_\nu &= \bigwedge_{\nu > \lambda} \bigwedge_{\mu > \nu} (E_\mu \vee F_\mu) \\ &= \bigwedge_{\mu > \lambda} (E_\mu \vee F_\mu) \\ &= (\mathfrak{E} \wedge \mathfrak{F})_\lambda. \qquad \Box \end{aligned}$$

At a first glance the infimum over μ in the definition of $1\mathfrak{E} \wedge \mathfrak{F}$ looks strange but it is necessary in order to guarantee the continuity from the right. This is shown by the following

Example 3.1 *Let $\mathcal{H}$ be separable, $(e_k)_{k \in \mathbb{N}}$ an orthonormal basis for $\mathcal{H}$, $x := \sum_{k=1}^{\infty} \frac{1}{k} e_k$, P the projection onto $\mathbb{C}x$ and P_n the projection onto*

$$U_n = \mathbb{C}e_1 + \ldots + \mathbb{C}e_n.$$

Note that $x \notin U_n$ for all $n \in \mathbb{N}$. Let $\mathfrak{E} = (E_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral family defined by

$$E_\lambda := \begin{cases} 0 & \text{for } \lambda \leq 0 \\ I - P_n & \text{for } \frac{1}{n+1} \leq \lambda < \frac{1}{n} \\ I & \text{for } \lambda \geq 1 \end{cases}$$

and let $\mathfrak{F}$ be the spectral family of P. Then

$$(\mathfrak{E} \wedge \mathfrak{F})_0 = \bigwedge_{\mu>0} (E_\mu \vee (I - P)).$$

As $x \notin U_n$ we obtain for $\mu \in [\frac{1}{n+1}, \frac{1}{n}[$:

$$\begin{aligned} E_\mu \vee (I - P) &= (I - P_n) \vee (I - P) \\ &= I - (P_n \wedge P) \\ &= I. \end{aligned}$$

Hence $(\mathfrak{E} \wedge \mathfrak{F})_0 = I$, but $E_0 \vee (I - P) = I - P < I$.

Remark 3.1 *If $\mathcal{R}$ is a finite von Neumann algebra and if $(P_\iota)_{\iota\in J}, (Q_\iota)_{\iota\in J}$ are decreasing nets (over the same index set J) in $\mathcal{P}(\mathcal{R})$ converging to projections P and Q respectively, then (see [7], p.412)*

$$\bigwedge_{\iota\in J} (P_\iota \vee Q_\iota) = P \vee Q.$$

Thus for finite $\mathcal{R}$ we have

$$(\mathfrak{E} \wedge \mathfrak{F})_\lambda = E_\lambda \vee F_\lambda \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

Definition 3.1 *Let $A, B \in \mathcal{R}_{sa}$ with corresponding spectral families E^A and E^B respectively. Then we define $A \wedge_s B$, $A \vee_s B$ as the operators in $\mathcal{R}_{sa}$ whose spectral families are $E^A \wedge E^B$ and $E^A \vee E^B$ respectively.*

Proposition 3.2 *$A \wedge_s B$ is the minimum and $A \vee_s B$ is the maximum of A and B in the sense of lattice theory.*

Proof: We have to check the universal properties of minimum and maximum. $A \wedge_s B \leq_s A$, for

$$\forall\ \lambda \in \mathbb{R}:\ E_\lambda^{A\wedge_s B} = \bigwedge_{\mu>\lambda} (E_\mu^A \vee E_\mu^B) \geq \bigwedge_{\mu>\lambda} E_\mu^A = E_\lambda^A$$

and similarly $A \wedge_s B \leq_s B$. If $C \in \mathcal{R}_{sa}$ such that $C \leq_s A, B$, then

$$E^A_\mu, E^B_\mu \leq E^C_\mu,$$

hence

$$E^A_\mu \vee E^B_\mu \leq E^C_\mu$$

for all $\mu \in \mathbb{R}$, and therefore

$$\bigwedge_{\mu > \lambda} (E^A_\mu \vee E^B_\mu) \leq \bigwedge_{\mu > \lambda} E^C_\mu = E^C_\lambda.$$

This shows $C \leq_s A \wedge_s B$. In the same way one can prove that $A, B \leq_s A \vee_s B$ and that $A, B \leq_s C$ implies $A \vee_s B \leq_s C$. $\square$

Thus $\mathcal{R}_{sa}$ together with the spectral order $\leq_s$ is a lattice which we call the **spectral lattice of** $\mathcal{R}$ and denote it by $(\mathcal{R}_{sa}, \leq_s)$. If we speak of the *lattice* $\mathcal{R}_{sa}$, we always mean this with respect to the spectral order. (There cannot be any confusion with the usual order: if $\mathcal{R}$ is not abelian, then $\mathcal{R}_{sa}$ is not a lattice with respect to $\leq$, and if $\mathcal{R}$ is abelian, then the two partial orders coincide.) From corollary 2.2 we obtain

Corollary 3.1 *The projection lattice $\mathcal{P}(\mathcal{R})$ is a sublattice of the spectral lattice $\mathcal{R}_{sa}$.*

Lemma 3.1 *For $A \in \mathcal{R}_{sa}$ let $[m(A), M(A)]$ be the smallest compact interval containing the spectrum $sp(A)$ ofA. Then for all $A, B \in \mathcal{R}_{sa}$*

$$\begin{aligned}
m(A \wedge_s B) &= min(m(A), m(B)),\\
M(A \wedge_s B) &\leq min(M(A), M(B)),\\
m(A \vee_s B) &\geq max(m(A), m(B)),\\
M(A \vee_s B) &= max(M(A), M(B)).
\end{aligned}$$

This is quite easy to see and so we omit the proof.

From this lemma and from lemma 2.1 we further obtain

Corollary 3.2 *For $a, b \in \mathbb{R}$, $a < b$, let*

$$\mathcal{R}_{[a,b]} := \{A \in \mathcal{R}_{sa} \mid aI \leq A \leq bI\}.$$

Then $(\mathcal{R}_{[a,b]}, \leq_s)$ is a sublattice of the spectral lattice $\mathcal{R}_{sa}$, isomorphic to $\mathcal{E}(\mathcal{R})$.

Theorem 3.1 *The spectral lattice $\mathcal{R}_{sa}$ is boundedly complete.*

Proof: Obviously, $\mathcal{R}_{sa}$ is a boundedly complete lattice if and only if $\mathcal{E}(\mathcal{R})$ is a complete lattice. We prove the completeness of $\mathcal{E}(\mathcal{R})$. Let $(A_\kappa)_\kappa \in \mathbb{K}$ be an arbitrary family in $\mathcal{E}(\mathcal{R})$ and let $(E^{A_\kappa})_{\kappa\in\mathbb{K}}$ be the corresponding family of spectral families. Let

$$\forall\ \lambda \in \mathbb{R}:\ E^{\vee}_\lambda := \bigwedge_{\kappa\in\mathbb{K}} E^{A_\kappa}_\lambda .$$

It is quite easy to check that $\mathfrak{E}^{\vee} := (E^{\vee}_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral family and that the corresponding operator $A_{\mathfrak{E}^{\vee}}$ belongs to $\mathcal{E}(\mathcal{R})$. From the definition of $\mathfrak{E}^{\vee}$ we have $A_\kappa \leq_s A_{\mathfrak{E}^{\vee}}$ for all $\kappa \in \mathbb{K}$. Let $B \in \mathcal{E}(\mathcal{R})$ with spectral family $\mathfrak{F} = (F_\lambda)_{\lambda\in\mathbb{R}}$ such that $A_\kappa \leq_s B$ for all $\kappa \in \mathbb{K}$, i.e.

$$\forall\ \kappa \in \mathbb{K}\ \forall\ \lambda \in \mathbb{R}:\ F_\lambda \leq E^{A_\kappa}_\lambda .$$

Hence

$$\forall\ \lambda \in \mathbb{R}:\ F_\lambda \leq \bigwedge_{\kappa\in\mathbb{K}} E^{A_\kappa}_\lambda ,$$

i.e. $A_{\mathfrak{E}^{\vee}} \leq_s B$. Therefore

$$\bigvee_{\kappa\in\mathbb{K}} A_\kappa := A_{\mathfrak{E}^{\vee}}$$

is the *supremum of the family* $(A_\kappa)_{\kappa\in\mathbb{K}}$.
In order to show that $(A_\kappa)_{\kappa\in\mathbb{K}}$ has an infimum we set

$$E^{\wedge}_\lambda := \bigwedge_{\mu>\lambda} \bigvee_{\kappa\in\mathbb{K}} E^{A_\kappa}_\mu .$$

We show that $\mathfrak{E}^{\wedge} := (E^{\wedge}_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral family.
The properties $E^{\wedge}_\lambda = 0$ for $\lambda < 0$ and $E^{\wedge}_\lambda = 1$ for $\lambda \geq 1$ are obvious.
Let $\lambda_1 < \lambda_2$ and μ, ν such that $\lambda_1 < \mu < \lambda_2 < \nu$. Then $\bigvee_\kappa E^{A_\kappa}_\mu \leq \bigvee_\kappa E^{A_\kappa}_\nu$ and therefore

$$\bigvee_{\kappa} E^{A_\kappa}_\mu \leq \bigwedge_{\nu>\lambda_2} \bigvee_{\kappa} E^{A_\kappa}_\nu .$$

This implies

$$\bigwedge_{\mu>\lambda_1} \bigvee_{\kappa} E^{A_\kappa}_\mu \leq \bigwedge_{\nu>\lambda_2} \bigvee_{\kappa} E^{A_\kappa}_\nu ,$$

i.e.

$$E^{\wedge}_{\lambda_1} \leq E^{\wedge}_{\lambda_2} .$$

Finally we have

$$\begin{aligned}\bigwedge_{\mu>\lambda} E_\mu^\wedge &= \bigwedge_{\mu>\lambda}\bigwedge_{\nu>\mu}\bigvee_\kappa E_\nu^{A_\kappa} \\ &= \bigwedge_{\mu>\lambda}\bigvee_\kappa E_\mu^{A_\kappa} \\ &= E_\lambda^\wedge.\end{aligned}$$

Hence $\mathfrak{E}^\wedge$ is a spectral family in $\mathcal{E}(\mathcal{R})$. Eventually we prove that the operator $A_{\mathfrak{E}^\wedge}$ corresponding to this spectral family is the infimum of $(A_\kappa)_{\kappa\in\mathbb{K}}$.
Let $B \in \mathcal{E}(\mathcal{R})$ with spectral family $\mathfrak{F} = (F_\lambda)_\lambda \in \mathbb{R}$ such that $B \leq A_\kappa$ for all $\kappa \in \mathbb{K}$. Then $\bigvee_\kappa E_\mu^{A_\kappa} \leq F_\mu$ for all μ, hence

$$\forall\ \nu > \lambda : \bigwedge_{\mu>\lambda}\bigvee_\kappa E_\mu^{A_\kappa} \leq F_\nu$$

and therefore

$$E_\lambda^\wedge = \bigwedge_{\mu>\lambda}\bigvee_\kappa E_\mu^{A_\kappa} \leq \bigwedge_{\nu>\lambda} F_\nu = F_\lambda$$

for all λ. Thus

$$B \leq A_{\mathfrak{E}^\wedge}. \qquad \square$$

4 Complements

We have defined the spectral order and the corresponding lattice operations in terms of spectral families. In the same way we proceed to define complementations.
If $\mathfrak{E} = (E_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral family in the von Neumann algebra $\mathcal{R}$ then $\lambda \mapsto I - E_{-\lambda}$ is increasing but it is not necessarily continuous from the right:

$$\begin{aligned}\bigwedge_{\mu>\lambda}(I - E_{-\mu}) &= \bigwedge_{\mu<-\lambda}(I - E_\mu) \\ &= I - \bigvee_{\mu<-\lambda} E_\mu \\ &= I - E_{-\lambda-0},\end{aligned}$$

where $E_{a-0} := \bigvee_{\mu<a} E_\mu$. This leads to the definition

$$(\neg\mathfrak{E})_\lambda := \bigwedge_{\mu>\lambda}(I - E_{-\mu}) = I - E_{-\lambda-0}.$$

It is easy to check that $\neg\mathfrak{E}$ is a spectral family. We call it the **free complement of $\mathfrak{E}$**.

Proposition 4.1 *Let $A \in \mathcal{R}_{sa}$ with spectral family E^A. Then $\neg E^A = E^{-A}$.*

Proof: The hermitean operator given by $\neg E^A$ is

$$\neg A := \int_{\mathbb{R}} \lambda d(\neg E^A)_\lambda.$$

From $(\neg E^A)_\lambda = I - E^A_{-\lambda-0}$ we obtain

$$\neg A = \int_{\mathbb{R}} \lambda d(-E^A_{-\lambda-0}).$$

If $\mathfrak{Z} = (\lambda_k)_{k\in K}$ is a partition of $\mathbb{R}$, then

$$\sum_k \mu_k(-E^A_{-\lambda_{k+1}-0} + E^A_{-\lambda_k-0}) = -\sum_k(-\mu_k)(E^A_{-\lambda_k-0} - E^A_{-\lambda_{k+1}-0})$$

where $\mu_k \in]\lambda_k, \lambda_{k+1}[$. This converges to $-A$ as the width $|\mathfrak{Z}|$ of $\mathfrak{Z}$ tends to zero because of

$$E^A([a,b[) = E^A_{b-0} - E^A_{a-0} \quad \text{and} \quad E^A(]a,b]) = E^A_b - E^A_a.$$

Hence $\neg A = -A$. $\square$

Corollary 4.1 *$\neg(\neg\mathfrak{E}) = \mathfrak{E}$ for all spectral families $\mathfrak{E}$ in $\mathcal{R}$.*

If $A \in \mathcal{R}_{sa}$ then obviously

$$[m(-A), M(-A)] = [-M(A), -m(A)],$$

so

Remark 4.1 *If $A \in \mathcal{R}_{[a,b]}$ then $(a+b)I - A \in \mathcal{R}_{[a,b]}$. Especially $I - A \in \mathcal{E}(\mathcal{R})$ for $A \in \mathcal{E}(\mathcal{R})$.*

If $A \in \mathcal{E}(\mathcal{R})$ then $I - A$ is called the **Kleene complement** ([1]). If A is a projection, then $A \wedge (I - A) = 0$. This is not true for general $A \in \mathcal{E}(\mathcal{R})$:

Proposition 4.2 *Let $A \in \mathcal{E}(\mathcal{R})$. Then $A \wedge (I - A) = 0$ if and only if A is a projection.*

This is a well known result with a quite simple proof: Consider $A, I - A$ as continuous functions $sp(A) \to [0,1]$. If $A(\lambda) > 0$ then $A \wedge (I - A) = 0$ implies $1 - A(\lambda) = 0$, i.e. $A(\lambda) = 1$. This means that $imA \subseteq \{0,1\}$, i.e. that A is a projection.

The Kleene complement satisfies the de Morgan rules in the lattice $\mathcal{E}(\mathcal{R})$:

Proposition 4.3 *Let $A, B \in \mathcal{E}(\mathcal{R})$. Then*

(i) $A \leq_s B$ *if and only if* $I - B \leq_s I - A$,

(ii) $I - (A \wedge_s B) = (I - A) \vee_s (I - B)$,

(iii) $I - (A \vee_s B) = (I - A) \wedge_s (I - B)$.

Proof: $A \leq_s B$ implies

$$E_\lambda^{I-A} = I - E^A_{(1-\lambda)-0} \leq I - E^B_{(1-\lambda)-0} = E_\lambda^{I-B}$$

for all λ, hence $I - B \leq_s I - A$.
From the universal property of the maximum we conclude that $I - (A \wedge_s B) = (I - A) \vee_s (I - B)$ if and only if

$$\forall\, C \in \mathcal{E}(\mathcal{R}): \ (I - A \leq_s C, I - B \leq_s C \Longrightarrow I - (A \wedge_s B) \leq_s C).$$

This follows from (i) and the universal property of the minimum:

$$\begin{aligned} I - A \leq_s C, I - B \leq_s C &\Longrightarrow I - C \leq_s A, I - C \leq_s B \\ &\Longrightarrow I - C \leq_s A \wedge_s B \\ &\Longrightarrow I - (A \wedge_s B) \leq_s C. \end{aligned}$$

(iii) follows from (i) and (ii). $\square$

Corollary 4.2 *Let $A \in \mathcal{E}(\mathcal{R})$. Then $A \vee (I - A) = I$ if and only if A is a projection.*

If $A \in \mathcal{R}$ then the projection onto the closure of imA is called the **range projection of** A and is usually denoted by $R(A)$. Obviously

$$R(A) = \bigwedge\{P \in \mathcal{P}(\mathcal{R}) |\ PA = A\}.$$

Lemma 4.1 *Let $A \in \mathcal{E}(\mathcal{R})$. Then*

$$R(A) = \bigwedge\{P \in \mathcal{P}(\mathcal{R}) |\ A \ \leq P\} = I - E_0^A.$$

Proof: If $A \in \mathcal{E}(\mathcal{R})$ and $P \in \mathcal{P}(\mathcal{R})$, then $PA = A$ implies

$$A = PA = PAP \leq P.$$

Conversely $A \leq P$ implies $A = PA$ by the proof of corollary 2.2. Hence $R(A) = \bigwedge\{P \in \mathcal{P}(\mathcal{R}) |\ A \leq P\}$.
If P is a projection in $\mathcal{R}$ then $A \leq P$ is equivalent to $A \leq_s P$. This is equivalent to $I - P \leq E_0^A$ i.e. to $I - E_0^A \leq P$. Therefore $R(A) = I - E_0^A$.
$\square$

Definition 4.1 *([1])* $A^{\sim} := I - R(A) = E_0^A$ *is called the* **Brouwer complement of** $A \in \mathcal{E}(\mathcal{R})$.

The Brouwer complement has the following (certainly well known) properties:

Proposition 4.4 *For all* $A \in \mathcal{E}(\mathcal{R})$ *the following properties hold:*

(i) $A^{\sim\sim} = R(A)$,

(ii) $A \wedge A^{\sim} = 0$, *but*

(iii) $A \vee A^{\sim} = I$ *if and only if* A *is a projection.*

Proof: (i) is obvious. $A \wedge (I - R(A)) \leq R(A) \wedge (I - R(A)) = 0$ gives (ii). (iii) follows from

$$E_\lambda^{A \vee A^{\sim}} = E_\lambda^A \wedge (I - E_0^A) = E_\lambda^A - E_0^A$$

for all $\lambda \in [0, 1[$. $\square$

Eventually we will show that the **hull operation** $R : \mathcal{E}(\mathcal{R}) \to \mathcal{P}(\mathcal{R})$ which sends A to its range projection $R(A)$ is a lattice homomorphism if and only if $\mathcal{R}$ is a finite von Neumann algebra.

Lemma 4.2 *For all* $A, B \in \mathcal{E}(\mathcal{R})$ *we have*

(i) $R(A \vee_s B) = R(A) \vee R(B)$,

(ii) $R(A \wedge_s B) \leq R(A) \wedge R(B)$.

Proof: Although these properties follow immediately from lemma 4.1 and the universal property of the lattice operations, it is instructive to give a proof that uses the definition of the lattice operations.
$E_0^{A \vee_s B} = E_0^A \wedge E_0^B$ implies

$$R(A \vee_s B) = I - E_0^A \wedge E_0^B = (I - E_0^A) \vee (I - E_0^B) = R(A) \vee R(B),$$

and

$$E_0^{A \wedge_s B} = \bigwedge_{\lambda > 0} (E_\lambda^A \vee E_\lambda^B) \geq E_0^A \vee E_0^B$$

implies

$$R(A \wedge_s B) = I - E_0^{A \wedge_s B} \leq I - (E_0^A \vee E_0^B) = (I - E_0^A) \wedge (I - E_0^B) = R(A) \wedge R(B). \ \square$$

Remark 4.2 *The proof gives the essential hint for proving that* $R(A \wedge_s B) = R(A) \wedge R(B)$ *for all* $A, B \in \mathcal{E}(\mathcal{R})$ *forces the finiteness of* $\mathcal{R}$. *Example 3.1 shows that in* $\mathcal{L}(\mathcal{H})$ *the strict inequality* $R(A \wedge_s B) < R(A) \wedge R(B)$ *can occur and we shall construct a similar example in any non-finite von Neumann algebra.*

Corollary 4.3 *The Brouwer complement satisfies the first de Morgan law*

$$\forall\ A, B \in \mathcal{E}(\mathcal{R}) : \ (A \vee_s B)^{\sim} = A^{\sim} \wedge B^{\sim}.$$

Lemma 4.3 *Let* $\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})$ *be a von Neumann algebra acting on a Hilbert space* $\mathcal{K}$ *with unity* $I_{\mathcal{M}} = id_{\mathcal{K}}$ *and let* $(E_\lambda)_{\lambda \in \mathbb{R}}$ *be a spectral family in the von Neumann algebra* $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$. *Then for all* $A, B \in \mathcal{R}_{sa}$ *and all* $P, Q \in \mathcal{P}(\mathcal{R})$*:*

(i) $I_{\mathcal{M}} \otimes A \leq I_{\mathcal{M}} \otimes B$ *if and only if* $A \leq B$.

(ii) $I_{\mathcal{M}} \otimes (P \wedge Q) = (I_{\mathcal{M}} \otimes P) \wedge (I_{\mathcal{M}} \otimes Q)$.

(iii) $I_{\mathcal{M}} \otimes (P \vee Q) = (I_{\mathcal{M}} \otimes P) \vee (I_{\mathcal{M}} \otimes Q)$.

(iv) $(I_{\mathcal{M}} \otimes E_\lambda)_{\lambda \in \mathbb{R}}$ *is a spectral family in* $\mathcal{M} \bar{\otimes} \mathcal{R}$.

Proof: We use some results on tensor products that can be found in [4, 5, 11]. Let $(e_b)_{b \in \mathbb{B}}$ be an orthonormal basis of $\mathcal{K}$. Then

$$U : \sum_{b \in \mathbb{B}} x_b \mapsto \sum_{b \in \mathbb{B}} (e_b \otimes x_b)$$

is a surjective isometry from $\bigoplus_{b \in \mathbb{B}} \mathcal{H}_b$ (with $\mathcal{H}_b = \mathcal{H}$ for all $b \in \mathbb{B}$) onto $\mathcal{K} \otimes \mathcal{H}$. Let $A \in \mathcal{R}$. U intertwines $I_{\mathcal{M}} \otimes A$ and A :

$$U^{-1}(I_{\mathcal{M}} \otimes A)U = \bigoplus_{b \in \mathbb{B}} A_b$$

with $A_b = A$ for all $b \in \mathbb{B}$. This immediately implies (i).
Note that $I_{\mathcal{M}} \otimes A$ is a projection if and only if A is. Then (ii) and (iii) follow from (i) and the universal property of minimum and maximum.
Let $(E_\lambda)_{\lambda \in \mathbb{R}}$ be a spectral family in $\mathcal{R}$. Then $\lambda \mapsto I_{\mathcal{M}} \otimes E_\lambda$ is monotonic increasing, equals $I_{\mathcal{M}} \otimes I$ for λ large enough and zero for λ small enough. In order to prove the continuity from the right we use the fact that the mapping $A \mapsto I_{\mathcal{M}} \otimes A$ from $\mathcal{R}$ to $\mathcal{M} \bar{\otimes} \mathcal{R}$ is strongly continuous on bounded subsets of $\mathcal{R}$:

$$\bigwedge_{\mu > \lambda} (I_{\mathcal{M}} \otimes E_\mu) = I_{\mathcal{M}} \otimes \bigwedge_{\mu > \lambda} E_\mu = I_{\mathcal{M}} \otimes E_\lambda.$$

Hence also (iv) follows. $\square$

Theorem 4.1 *The mapping $R : \mathcal{E}(\mathcal{R}) \to \mathcal{P}(\mathcal{R}),\ \ A \mapsto R(A)$ is a homomorphism of lattices if and only if $\mathcal{R}$ is a finite von Neumann algebra.*

Proof: Remark 3.1 shows that $R : \mathcal{E}(\mathcal{R}) \to \mathcal{P}(\mathcal{R})$ is a lattice homomorphism if $\mathcal{R}$ is finite. Now assume that $\mathcal{R}$ is not finite. Then $\mathcal{R}$ contains a direct summand of the form $\mathcal{M}\bar{\otimes}\mathcal{L}(\mathcal{H}_0)$, where $\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})$ is a suitable von Neumann algebra and $\mathcal{H}_0$ a separable Hilbert space of infinite dimension (see e.g. [11], Ch. V.1, essentially prop. 1.22: if $\mathcal{R}$ is not finite then $\mathcal{R}$ has a direct summand with properly infinite unity I_0. Use the halving lemma to construct a countable infinite orthogonal sequence of pairwise equivalent projections with sum I_0 (see the proof of theorem 6.3.4 in [5])). Take the spectral families $(E_\lambda)_{\lambda\in\mathbb{R}}$ and E^P in $\mathcal{L}(\mathcal{H}_0)$ we have defined in example 3.1. Then by lemma 4.3

$$\begin{aligned}
\bigwedge_{\mu>0}((I_{\mathcal{M}} \otimes E_\mu) \vee (I_{\mathcal{M}} \otimes (I-P))) &= \bigwedge_{\mu>0}(I_{\mathcal{M}} \otimes (E_\mu \vee (I-P))) \\
&= I_{\mathcal{M}} \otimes \bigwedge_{\mu>0}(E_\mu \vee (I-P)) \\
&> I_{\mathcal{M}} \otimes (E_0 \vee (I-P)).
\end{aligned}$$

Therefore we obtain for the corresponding operators $I_{\mathcal{M}} \otimes A$ and $I_{\mathcal{M}} \otimes P$:

$$R((I_{\mathcal{M}} \otimes A) \wedge_s (I_{\mathcal{M}} \otimes P)) < R(I_{\mathcal{M}} \otimes A) \wedge R(I_{\mathcal{M}} \otimes P),$$

i.e. R is not a lattice homomorphism. $\square$

Corollary 4.4 *The von Neumann algebra $\mathcal{R}$ is finite if and only if the second de Morgan law*

$$(A \wedge_s B)^{\sim} = A^{\sim} \vee B^{\sim}$$

for the Brouwer complement is satisfied for all $A, B \in \mathcal{E}(\mathcal{R})$.

Acknowledgements. Sincere thanks go to Andreas Döring for several discussions on the subject and for bringing the paper of C.Cattaneo and J.Hamhalter to my attention. Thanks go also to David Sherman (Santa Barbara) for informing me about Olson's paper.

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